

SYMMETRY OF A SYMPLECTIC TORIC MANIFOLD

MIKIYA MASUDA

ABSTRACT. The action of a torus group T on a symplectic toric manifold (M, ω) often extends to an effective action of a (non-abelian) compact Lie group G . We may think of T and G as compact Lie subgroups of the symplectomorphism group $\text{Symp}(M, \omega)$ of (M, ω) . On the other hand, (M, ω) is determined by the associated moment polytope P by the result of Delzant [4]. Therefore, the group G should be estimated in terms of P or we may say that a *maximal* compact Lie subgroup of $\text{Symp}(M, \omega)$ containing the torus T should be described in terms of P .

In this paper, we introduce a root system $R(P)$ associated to P and prove that any irreducible subsystem of $R(P)$ is of type A and the root system $\Delta(G)$ of the group G is a subsystem of $R(P)$ (so that $R(P)$ gives an upper bound for the identity component of G and any irreducible factor of $\Delta(G)$ is of type A). We also introduce a homomorphism \mathcal{D} from the normalizer $N_G(T)$ of T in G to an automorphism group $\text{Aut}(P)$ of P , which detects the connected components of G . Finally we find a maximal compact Lie subgroup G_{\max} of $\text{Symp}(M, \omega)$ containing the torus T .

1. INTRODUCTION

A *symplectic toric manifold* is a compact connected symplectic manifold (M, ω) with an effective Hamiltonian action of a torus group T of half the dimension of the manifold M . Delzant [4] proves that M is equivariantly diffeomorphic to a smooth projective toric variety with the restricted T -action. Moreover he classifies symplectic toric manifolds by showing that the correspondence from symplectic toric manifolds to their moment polytopes is one-to-one. Therefore, all geometrical information on (M, ω) is encoded in the moment polytope P associated with (M, ω) .

Date: June 24, 2009.

2000 *Mathematics Subject Classification.* Primary 53D20, 57S15; Secondary 14M25.

Key words and phrases. symplectic toric manifold, moment polytope, root system.

The author was partially supported by Grant-in-Aid for Scientific Research 19204007.

The T -action on (M, ω) often extends to an effective action of a (non-abelian) compact Lie group G . We may think of T and G as compact Lie subgroups of the symplectomorphism group $\text{Symp}(M, \omega)$ of (M, ω) . Since the T -fixed point set in M is non-empty and $\dim T = \frac{1}{2} \dim M$, there is no torus subgroup of $\text{Symp}(M, \omega)$ containing T properly. This means that T is a maximal torus of G .

In this paper, we introduce a root system $R(P)$ associated to the moment polytope P and a homomorphism

$$(1.1) \quad \mathcal{D}: N_G(T) \rightarrow \text{Aut}(P),$$

where $N_G(T)$ denotes the normalizer of T in G and $\text{Aut}(P)$ denotes an automorphism group of P . It turns out that the root system $R(P)$ gives information on the identity component G^0 of G and the homomorphism \mathcal{D} induces an injective homomorphism

$$G/G^0 \cong N_G(T)/N_{G^0}(T) \rightarrow \text{Aut}(P)/\mathcal{D}(N_{G^0}(T))$$

so that \mathcal{D} detects the connected components of G . Here is a summary of our results.

Theorem 1.1. *Let (M, ω) be a symplectic toric manifold with a Hamiltonian action of a torus T of half the dimension of the dimension of M and let P be the associated moment polytope. Then the following hold.*

- (1) *Any irreducible subsystem of $R(P)$ is of type A .*
- (2) *If G is a compact Lie subgroup of $\text{Symp}(M, \omega)$ containing the torus T , then the root system $\Delta(G)$ of G is a subsystem of $R(P)$, so that any irreducible factor of $\Delta(G)$ is of type A by (1) above.*
- (3) *Let G be as in (2) above. If $\Delta(G) = R(P)$ and the homomorphism \mathcal{D} in (1.1) is surjective, then G is maximal among compact Lie subgroups of $\text{Symp}(M, \omega)$, i.e., G is not properly contained in a compact Lie subgroup of $\text{Symp}(M, \omega)$.*
- (4) *There exists a compact Lie subgroup G_{\max} of $\text{Symp}(M, \omega)$ containing the torus T such that the assumption in (3) above is satisfied.*

Remark. By (3) above, the group G_{\max} in (4) above, which contains the torus T , is maximal among compact Lie subgroups of $\text{Symp}(M, \omega)$. However, the author does not know whether any maximal compact Lie subgroup of $\text{Symp}(M, \omega)$ containing T is conjugate to G_{\max} in $\text{Symp}(M, \omega)$, where the torus T is fixed. Related to this question, it is proved in [8] that when $\dim M = 4$, the number of conjugacy classes of 2-dimensional tori in $\text{Symp}(M, \omega)$ is finite.

Our work is motivated by the work of Demazure [5] (see also [3] or [11, Section 3.5]). He introduces a root system $R(\Delta)$ for a complete

non-singular fan Δ and proves that it agrees with the root system of the automorphism group $\text{Aut}(X(\Delta))$ of the compact smooth toric variety $X(\Delta)$ associated with Δ , where $\text{Aut}(X(\Delta))$ is known to be an algebraic group, and that $R_s(\Delta) := R(\Delta) \cap (-R(\Delta))$ is the root system of the reductive (or semisimple) part of $\text{Aut}(X(\Delta))$. The symplectic toric manifold (M, ω) is equivariantly diffeomorphic to a smooth projective toric variety X with the restricted T -action as mentioned before and the fan Δ_X of the X is the so-called normal fan derived from the moment polytope P associated with (M, ω) , where normal vectors v_i 's to facets of P are edge vectors in the fan Δ_X . One sees that our root system $R(P)$ agrees with $R_s(\Delta_X)$. Demazure also describes the connected components of $\text{Aut}(X(\Delta))$ in terms of the automorphism group $\text{Aut}(\Delta)$ of the fan Δ . The automorphism group $\text{Aut}(P)$ of P can be regarded as a counterpart to $\text{Aut}(\Delta)$, in fact, $\text{Aut}(P)$ can be regarded as a subgroup of $\text{Aut}(\Delta)$. We remark that the root systems $R(P)$ and $R(\Delta)$ depend only on the vectors v_i 's but $\text{Aut}(P)$ and $\text{Aut}(\Delta)$ are not determined by the vectors.

This paper is organized as follows. In Section 2 we review an explicit construction (called the *Delzant construction* in [6]) of a symplectic toric manifold (M, ω) with moment polytope P . In Section 3 we rewrite the construction in terms of equivariant (co)homology and also recall some facts on the equivariant cohomology of M . In Section 4 we make some observations on roots of a compact Lie subgroup G of $\text{Symp}(M, \omega)$ containing the torus T . Based on the observations, we introduce the root system $R(P)$ and prove the assertions (1) and (2) in Theorem 1.1 (see Proposition 5.1, Theorem 5.4 and Corollary 5.5). In Section 6 we find a *connected* compact Lie subgroup G of $\text{Symp}(M, \omega)$ which attains the equality $\Delta(G) = R(P)$. In Section 7 we introduce the homomorphism \mathcal{D} in (1.1) for an arbitrary subgroup G of $\text{Symp}(M, \omega)$ containing T and observe that \mathcal{D} detects the connected components of G when G is a compact Lie group. The assertions (3) and (4) in Theorem 1.1 are proved in Section 8 (see Theorem 8.1).

Throughout this paper, (M, ω) will denote a symplectic toric manifold with moment polytope P , where a Hamiltonian T -action on (M, ω) is incorporated although it is often not mentioned explicitly. The argument developed in this paper works for *torus manifolds* introduced in [7] with some modification. We will discuss this in a forthcoming paper.

2. DELZANT CONSTRUCTION

By the result of Delzant mentioned in the Introduction, a symplectic toric manifold (M, ω) is determined by the associated moment polytope P and is explicitly constructed from P . We will review the construction in this section. The details can be found in [6].

Let $\mu: M \rightarrow \mathfrak{t}^*$ be a moment map associated with (M, ω) so that $\mu(M) = P$, where \mathfrak{t}^* is the dual of the Lie algebra \mathfrak{t} of T . The moment map μ is uniquely determined by (M, ω) up to parallel translations in \mathfrak{t}^* . We identify \mathfrak{t} with \mathbb{R}^n and \mathfrak{t}^* with $(\mathbb{R}^n)^*$ and express

$$(2.1) \quad P = \{u \in (\mathbb{R}^n)^* \mid \langle u, v_i \rangle \geq a_i \quad (i = 1, \dots, m)\}$$

where $v_i \in \mathbb{Z}^n$ is primitive, $\langle \cdot, \cdot \rangle$ is a natural pairing (i.e. evaluation) and $a_i \in \mathbb{R}$. Without loss of generality we may assume that there is no redundant inequality in (2.1) so that the intersection of P with the hyperplane defined by $\langle u, v_i \rangle = a_i$ is a facet (i.e. codimension 1 face) of P for each i , which we denote by P_i . So there are exactly m facets in P . The moment polytope P is *non-singular*, which means that P is simple and whenever n facets of P meet at a vertex, the n vectors v_i 's normal to the n facets form a basis of \mathbb{Z}^n . A non-singular polytope is called a *Delzant polytope* in [6].

Let e_1, \dots, e_m be the standard basis of \mathbb{Z}^m and consider the linear map $\pi_*: \mathbb{Z}^m \rightarrow \mathbb{Z}^n$ sending e_i to v_i for $i = 1, \dots, m$. Since P is non-singular, π_* is surjective and we have an exact sequence

$$(2.2) \quad 0 \rightarrow \text{Ker } \pi_* \xrightarrow{\iota_*} \mathbb{Z}^m \xrightarrow{\pi_*} \mathbb{Z}^n \rightarrow 0$$

where ι_* is the inclusion. Taking the dual of this sequence, we obtain an exact sequence

$$(2.3) \quad 0 \leftarrow (\text{Ker } \pi_*)^* \xleftarrow{\iota^*} (\mathbb{Z}^m)^* \xleftarrow{\pi^*} (\mathbb{Z}^n)^* \leftarrow 0$$

and one easily sees that

$$(2.4) \quad \pi^*(u) = \sum_{i=1}^m \langle u, v_i \rangle e_i^* \quad \text{for any } u \in (\mathbb{Z}^n)^*$$

where e_1^*, \dots, e_m^* denote the dual basis of e_1, \dots, e_m .

The map π_* (resp. π^*) extends to a linear map from \mathbb{R}^m onto \mathbb{R}^n (resp. from $(\mathbb{R}^n)^*$ to $(\mathbb{R}^m)^*$) and we use the same notation for the extended map. We define $\pi_a^*: (\mathbb{R}^n)^* \rightarrow (\mathbb{R}^m)^*$ by

$$\pi_a^*(u) := \pi^*(u) - \sum_{i=1}^m a_i e_i^* = \sum_{i=1}^m (\langle u, v_i \rangle - a_i) e_i^*.$$

The map π_a^* embeds P into the positive orthant of $(\mathbb{R}^m)^*$. The fiber product of π_a^* and the (moment) map

$$(2.5) \quad \Phi: \mathbb{C}^m \rightarrow (\mathbb{R}^m)^*$$

sending $z = (z_1, \dots, z_m)$ to $\frac{1}{2} \sum_{i=1}^m |z_i|^2 e_i^*$ is

$$(2.6) \quad \{(z, u) \in \mathbb{C}^m \times (\mathbb{R}^n)^* \mid \frac{1}{2} \sum_{i=1}^m |z_i|^2 e_i^* = \pi_a^*(u)\}.$$

Since π_a^* is injective and $\iota^*(\pi_a^*(u)) = -\sum_{i=1}^m a_i \iota^*(e_i^*)$ for any u , the first projection from $\mathbb{C}^m \times (\mathbb{R}^n)^*$ onto \mathbb{C}^m maps the fiber product (2.6) diffeomorphically onto

$$(2.7) \quad \mathcal{Z}_P := \{z \in \mathbb{C}^m \mid \sum_{i=1}^m (\frac{1}{2}|z_i|^2 + a_i) \iota^*(e_i^*) = 0\}.$$

Note that

$$(2.8) \quad \mathcal{Z}_P = \Phi^{-1}(\pi_a^*(P)).$$

Remark. The manifold \mathcal{Z}_P is called the *moment-angle manifold* of P and its topology is intensively studied in [2]. It is also studied in [1] from the viewpoint of real algebraic geometry.

We identify \mathbb{R}/\mathbb{Z} with the unit circle S^1 of the complex numbers \mathbb{C} through the exponential map $x \rightarrow \exp(2\pi\sqrt{-1}x)$ and set

$$T := (S^1)^n.$$

The map π_* in (2.2) induces an epimorphism

$$\mathcal{V}: (S^1)^m \rightarrow T$$

and we make an identification

$$(2.9) \quad (S^1)^m / \text{Ker } \mathcal{V} = T$$

through the map \mathcal{V} . An element $c = (c_1, \dots, c_n) \in \mathbb{Z}^n$ defines a homomorphism

$$(2.10) \quad \lambda_c: S^1 \rightarrow T$$

sending g to $(g^{c_1}, \dots, g^{c_n})$ and we note that

$$(2.11) \quad \mathcal{V}(g_1, \dots, g_m) = \prod_{i=1}^m \lambda_{v_i}(g_i) \quad \text{for } (g_1, \dots, g_m) \in (S^1)^m.$$

An element $b = (b_1, \dots, b_n)$ in \mathbb{Z}^n also defines a homomorphism

$$\chi^b: T \rightarrow S^1$$

sending (h_1, \dots, h_n) to $\prod_{i=1}^n h_i^{b_i}$. Then we have

$$(2.12) \quad (\chi^b \circ \lambda_c)(g) = g^{\langle b, c \rangle} \quad \text{for } g \in S^1$$

where $\langle b, c \rangle = \sum_{i=1}^n b_i c_i$. Since the intersection of the kernels of $\chi^b: T \rightarrow S^1$ for all b is trivial, it follows from (2.11) and (2.12) that

$$(2.13) \quad \text{Ker } \mathcal{V} = \{(g_1, \dots, g_m) \in (S^1)^m \mid \prod_{i=1}^m g_i^{\langle u, v_i \rangle} = 1 \text{ for } \forall u \in \mathbb{Z}^n\}.$$

Remark. The elements b and u above are taken from \mathbb{Z}^n but we will see that it would be better to regard them as elements of $(\mathbb{Z}^n)^*$ through the product $\langle \cdot, \cdot \rangle$.

The group $(S^1)^m$ acts on \mathbb{C}^m by componentwise multiplication and this action leaves \mathcal{Z}_P invariant. The map Φ in (2.5) induces a homeomorphism from the quotient $\mathbb{C}^m / (S^1)^m$ onto the positive orthant of $(\mathbb{R}^m)^*$ and $\Phi(\mathcal{Z}_P) = \pi_a^*(P)$. The action of $(S^1)^m$ restricted to $\text{Ker } \mathcal{V}$ is free on \mathcal{Z}_P and the quotient $\mathcal{Z}_P / \text{Ker } \mathcal{V}$ is known to be the given M . The standard symplectic form

$$\omega_0 := \frac{\sqrt{-1}}{2} \sum_{i=1}^m dz_i \wedge d\bar{z}_i$$

on \mathbb{C}^m is invariant under the linear action of the unitary group $U(m)$. The form ω_0 descends to the given ω on M . In fact, if

$$q: \mathcal{Z}_P \rightarrow M = \mathcal{Z}_P / \text{Ker } \mathcal{V}$$

denotes the quotient map, then ω satisfies

$$(2.14) \quad \omega_0|_{\mathcal{Z}_P} = q^*(\omega)$$

and is uniquely determined by this identity, where the left-hand side denotes the restriction of ω_0 to \mathcal{Z}_P . The action of $(S^1)^m$ on \mathcal{Z}_P induces an action of $T = (S^1)^m / \text{Ker } \mathcal{V}$ on $M = \mathcal{Z}_P / \text{Ker } \mathcal{V}$ and this T -action on M preserves the symplectic form ω .

As remarked before, the equation $\langle u, v_i \rangle = a_i$ defines the facet P_i of P for each $i = 1, \dots, m$, and

$$\mathcal{Z}_{P_i} := \Phi^{-1}(\pi_a^*(P_i)) \quad \text{and} \quad M_i := q(\mathcal{Z}_{P_i})$$

are respectively closed smooth submanifolds of \mathcal{Z}_P and M of real codimension 2. We call M_i 's the *characteristic submanifolds* of M . We see from (2.7) or (2.8) that

$$\mathcal{Z}_{P_i} = \mathcal{Z}_P \cap \{z_i = 0\}$$

and hence it follows from (2.10) and (2.11) that

Lemma 2.1. *The characteristic submanifold M_i is fixed pointwise by the S^1 -subgroup $\lambda_{v_i}(S^1)$ of T for each i .*

3. EQUIVARIANT COHOMOLOGY

It is more convenient and natural to interpret the Delzant construction in terms of equivariant (co)homology. We will discuss it and also recall some facts on equivariant cohomology in this section. Recall that the equivariant homology and cohomology of a space X with an action of the torus T are respectively defined as

$$H_*^T(X) := H_*(ET \times_T X) \quad \text{and} \quad H_T^*(X) := H^*(ET \times_T X)$$

where $ET \rightarrow BT = ET/T$ is a universal principal T -bundle and $ET \times_T X$ is the quotient of $ET \times X$ by the T -action given by

$$(3.1) \quad t(e, x) = (et^{-1}, tx) \quad \text{for } (e, x) \in ET \times X \text{ and } t \in T.$$

Let (M, ω) be a symplectic toric manifold and let M_i 's ($i = 1, \dots, m$) be the characteristic submanifolds of M . Since the T -action on M preserves the symplectic form ω and M_i is fixed pointwise under a circle subgroup of T (see Lemma 2.1), the ω restricted to M_i is again a symplectic form. Therefore the form ω and its restriction to M_i define orientations on M and M_i . Since M and M_i are oriented and the inclusion map from M_i to M is equivariant, it defines an equivariant Gysin homomorphism

$$H_T^*(M_i) \rightarrow H_T^{*+2}(M)$$

which raises the cohomological degree by 2 because the codimension of M_i in M is 2. We denote by τ_i the image of the unit element $1 \in H_T^0(M_i)$ by the equivariant Gysin homomorphism. The cohomological degree of τ_i is 2. We may think of τ_i as the Poincaré dual of the cycle M_i in the equivariant setting. Since a cup product $\prod_{i \in I} \tau_i$ for $I \subset [m] := \{1, \dots, m\}$ is the Poincaré dual of $\cap_{i \in I} M_i$, we see that

$$(3.2) \quad \prod_{i \in I} \tau_i = 0 \quad \text{if } \cap_{i \in I} M_i = \emptyset.$$

It turns out that $H_T^*(M)$ is generated by τ_i 's as a ring and that the relations in (3.2) are the only relations among τ_i 's, i.e. we have

Lemma 3.1. $H_T^*(M) = \mathbb{Z}[\tau_1, \dots, \tau_m] / (\prod_{i \in I} \tau_i \mid \cap_{i \in I} M_i = \emptyset)$ as rings.

In particular, τ_i 's are a free additive basis of $H_T^2(M)$ and the following easily follows from this fact.

Lemma 3.2. (see [10, Lemma 1.5] for example). *Let $\pi: ET \times_T M \rightarrow BT$ be the projection on the first factor. Then for each $i = 1, \dots, m$, there is a unique element $v_i \in H_2(BT)$ such that*

$$(3.3) \quad \pi^*(u) = \sum_{i=1}^m \langle u, v_i \rangle \tau_i \quad \text{for any } u \in H^2(BT)$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between cohomology and homology.

The Leray-Serre spectral sequence of the fibration

$$(3.4) \quad M \xrightarrow{\iota} ET \times_T M \xrightarrow{\pi} BT$$

collapses because $H^{odd}(M) = H^{odd}(BT) = 0$. Therefore $H_T^*(M) = H^*(BT) \otimes H^*(M)$ as $H^*(BT)$ -modules and hence $H^*(M)$ is the quotient of $H_T^*(M)$ by the ideal generated by $\pi^*(u)$ for $u \in H^2(BT)$. This together with Lemmas 3.1 and 3.2 implies the following well-known fact.

Proposition 3.3. *We set*

$$(3.5) \quad \mu_i := \iota^*(\tau_i) \in H^2(M).$$

Then $H^(M)$ is the quotient of a polynomial ring $\mathbb{Z}[\mu_1, \dots, \mu_m]$ by the ideal generated by the following two types of elements:*

- (1) $\prod_{i \in I} \mu_i$ for $I \subset [m]$ with $\cap_{i \in I} M_i = \emptyset$.
- (2) $\sum_{i=1}^m \langle u, v_i \rangle \mu_i$ for $u \in H^2(BT)$.

A homomorphism $f: S^1 \rightarrow T$ induces a continuous map $Bf: BS^1 \rightarrow BT$. We fix a generator κ of $H_2(BS^1) \cong \mathbb{Z}$. Then the correspondence $f \rightarrow (Bf)_*(\kappa)$ defines an isomorphism

$$(3.6) \quad \text{Hom}(S^1, T) \cong H_2(BT)$$

and we denote by λ_v the element of $\text{Hom}(S^1, T)$ corresponding to $v \in H_2(BT)$. The identity in Lemma 3.2 implies the following.

Lemma 3.4. (see [10, Lemma 1.10] for example). *For the elements $v_i \in H_2(BT)$ ($i = 1, \dots, m$) defined in Lemma 3.2, $\lambda_{v_i}(S^1)$ is the circle subgroup of T which fixes M_i pointwise.*

This lemma corresponds to Lemma 2.1. More precisely, one can see that the v_i 's defined in Lemma 3.2 can be identified with the v_i 's in Section 2 through an identification

$$H_2(BT) = \mathbb{Z}^n,$$

(see [10] for example). Taking the dual of this identification, we obtain an identification

$$H^2(BT) = (\mathbb{Z}^n)^*.$$

Then (2.1) can be rewritten as

$$(3.7) \quad P = \{u \in H^2(BT; \mathbb{R}) \mid \langle u, v_i \rangle \geq a_i \quad (i = 1, \dots, m)\}$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between cohomology and homology as before.

Lemma 3.5. *The exact sequences in (2.2) and (2.3) can be regarded as exact sequences derived from the fibration (3.4), namely, (2.2) can be regarded as*

$$(3.8) \quad 0 \rightarrow H_2(M) \xrightarrow{\iota_*} H_2^T(M) \xrightarrow{\pi_*} H_2(BT) \rightarrow 0$$

and (2.3) as

$$(3.9) \quad 0 \leftarrow H^2(M) \xleftarrow{\iota^*} H_T^2(M) \xleftarrow{\pi^*} H^2(BT) \leftarrow 0.$$

Proof. If we identify $H_T^2(M)$ with $(\mathbb{Z}^m)^*$ through the identification of τ_i with e_i^* for $i = 1, \dots, m$, then (3.3) agrees with (2.4) and this implies the lemma. \square

Through the identifications in Lemma 3.5, (2.7) turns into

$$(3.10) \quad \mathcal{Z}_P = \{z \in \mathbb{C}^m \mid \sum_{i=1}^m (\frac{1}{2}|z_i|^2 + a_i)\mu_i = 0\}$$

where μ_i 's are the elements of $H^2(M)$ defined in (3.5).

4. ROOTS OF A COMPACT LIE SUBGROUP OF $\text{Symp}(M, \omega)$

If $g \in \text{Symp}(M, \omega)$ normalizes the torus T , then ρ_g defined by

$$(4.1) \quad \rho_g(t) := gtg^{-1}$$

is a group automorphism of T and the diffeomorphism g of M is ρ_g -equivariant. Let $E\rho_g$ be a homeomorphism of ET induced from ρ_g . It is ρ_g -equivariant, i.e. $E\rho_g(et) = E\rho_g(e)\rho_g(t)$ for $e \in ET$ and $t \in T$. Therefore, a homeomorphism of $ET \times M$ sending (e, x) to $(E\rho_g(e), gx)$ is ρ_g -equivariant and induces a homeomorphism of $ET \times_T M$. Hence we obtain a ring automorphism of $H_T^*(M)$, denoted by g^* , which preserves the subalgebra $\pi^*(H^*(BT))$. It easily follows from the definition of g^* that

$$(4.2) \quad g^* \circ \pi^* = \pi^* \circ \rho_g^* \quad \text{on } H^*(BT)$$

where ρ_g^* is an automorphism of $H^*(BT)$ induced from ρ_g .

Since the diffeomorphism g of M is ρ_g -equivariant, it permutes the characteristic submanifolds M_i 's. Moreover, since g preserves the form ω , it preserves the orientations on M and M_i 's induced from ω . These imply that there is a permutation σ on $[m]$ such that

$$(4.3) \quad g^*(\tau_i) = \tau_{\sigma(i)} \quad \text{for any } i.$$

With these understood

Lemma 4.1. *Let ρ_{g*} be an automorphism of $H_*(BT)$ induced from ρ_g . Then $\rho_{g*}(v_{\sigma(i)}) = v_i$ for any i .*

Proof. Sending the identity (3.3) by g^* , it follows from (4.2) and (4.3) that we have

$$(4.4) \quad \pi^*(\rho_g^*(u)) = g^*(\pi^*(u)) = \sum_{i=1}^m \langle u, v_i \rangle g^*(\tau_i) = \sum_{i=1}^m \langle u, v_i \rangle \tau_{\sigma(i)},$$

while it follows from (3.3) applied to $\rho_g^*(u)$ instead of u that we have

$$(4.5) \quad \begin{aligned} \pi^*(\rho_g^*(u)) &= \sum_{i=1}^m \langle \rho_g^*(u), v_i \rangle \tau_i = \sum_{i=1}^m \langle \rho_g^*(u), v_{\sigma(i)} \rangle \tau_{\sigma(i)} \\ &= \sum_{i=1}^m \langle u, \rho_{g*}(v_{\sigma(i)}) \rangle \tau_{\sigma(i)}. \end{aligned}$$

Comparing (4.4) with (4.5) and noting that $\tau_{\sigma(i)}$'s are free over \mathbb{Z} , we obtain

$$\langle u, v_i \rangle = \langle u, \rho_{g*}(v_{\sigma(i)}) \rangle \quad \text{for any } i,$$

but since this identity holds for any $u \in H^2(BT)$, the desired identity in the lemma follows. \square

The following lemma is due to M. Wiemeler and will play a key role in the subsequent argument.

Lemma 4.2. ([12, Lemma 2.1]). *If g induces the identity on $H^2(M)$ and ρ_g^* is a reflection on $H^2(BT)$, then the σ in (4.3) permutes exactly two elements in $[m]$ and fixes the others.*

Proof. Since ρ_g^* is a reflection, its trace is $n - 2$. On the other hand, since g induces the identity on $H^2(M)$ by assumption, the trace of g^* on $H_T^2(M)$ must be $m - 2$ by (3.9). However, $H_T^2(M)$ is freely generated by τ_i 's over \mathbb{Z} and g^* permutes the generators by (4.3), so the lemma follows. \square

Dualizing the isomorphism (3.6), we obtain an isomorphism

$$(4.6) \quad \text{Hom}(T, S^1) \cong H^2(BT).$$

For $u \in H^2(BT)$, we denote by $\chi^u \in \text{Hom}(T, S^1)$ the element corresponding to u through the isomorphism (4.6).

Now we take a compact Lie subgroup G of $\text{Symp}(M, \omega)$ containing T and denote by G^0 the identity component of G . As remarked in the Introduction, the torus T is a maximal torus of G .

Definition. A root of G is a non-zero weight of the adjoint representation of T on $\mathfrak{g} \otimes \mathbb{C}$, where \mathfrak{g} denotes the Lie algebra of G . We think of a root of G as an element of $H^2(BT)$ through the isomorphism (4.6) and denote by $\Delta(G)$ the root system of G , that is the set of roots of G . Needless to say, $\Delta(G)$ depends only on the identity component G^0 .

For $\alpha \in \Delta(G)$, we denote by T_α the identity component of the kernel of $\chi^\alpha: T \rightarrow S^1$. Since α is non-zero, T_α is a codimension 1 subtorus of T . Let G_α^0 be the identity component of the subgroup of G^0 which commutes with T_α . The group $N_{G_\alpha^0}(T)/T$ is of order two and let $g \in N_{G_\alpha^0}(T)$ be a representative of the non-trivial element in $N_{G_\alpha^0}(T)/T$. The automorphism ρ_g of T is independent of the choice of the representative g , so we may denote it by ρ_α . It is of order two, its fixed point set contains the codimension 1 subtorus T_α and $\rho_\alpha^*(\alpha) = -\alpha$. We note that ρ_α^* is the Weyl group action associated with $\alpha \in \Delta(G)$.

Similarly, $\rho_{\alpha*}$ is a reflection on $H_2(BT)$ and we note that

$$(4.7) \quad \text{Fix}(\rho_{\alpha*}) = H_2(BT_\alpha) = \text{Ker } \alpha.$$

Lemma 4.3. *For $\alpha \in \Delta(G)$, there are $i, j \in [m]$ such that*

$$(4.8) \quad \langle \alpha, v_i \rangle = -\langle \alpha, v_j \rangle (\neq 0) \quad \text{and} \quad \langle \alpha, v_k \rangle = 0 \quad \text{for any } k \neq i, j.$$

Proof. Let $g \in N_{G_\alpha^0}(T)$ be a representative of the non-trivial element in $N_{G_\alpha^0}(T)/T$. Since g is in G^0 , it is homotopic to the identity so that g induces the identity on $H^2(M)$. Moreover, $\rho_g^* = \rho_\alpha^*$ is a reflection as observed above. Therefore there are $i, j \in [m]$ such that

$$g^*(\tau_i) = \tau_j, \quad g^*(\tau_j) = \tau_i, \quad g^*(\tau_k) = \tau_k \quad \text{for any } k \neq i, j$$

by Lemma 4.2. It follows from Lemma 4.1 that

$$(4.9) \quad \rho_{\alpha*}(v_i) = v_j, \quad \rho_{\alpha*}(v_j) = v_i, \quad \rho_{\alpha*}(v_k) = v_k \quad \text{for any } k \neq i, j$$

and hence $\langle \alpha, v_k \rangle = 0$ for $k \neq i, j$ by (4.7). Finally, since $\rho_\alpha^*(\alpha) = -\alpha$, we have

$$\langle \alpha, v_i \rangle = -\langle \rho_\alpha^*(\alpha), v_i \rangle = -\langle \alpha, \rho_{\alpha*}(v_i) \rangle = -\langle \alpha, v_j \rangle,$$

proving the lemma. \square

Lemma 4.4. *For $\alpha, \beta \in \Delta(G)$ we have*

$$\rho_\alpha^*(\beta) = \beta - \frac{\langle \beta, v_i \rangle - \langle \beta, v_j \rangle}{\langle \alpha, v_i \rangle} \alpha = \beta - \frac{\langle \beta, v_j \rangle - \langle \beta, v_i \rangle}{\langle \alpha, v_j \rangle} \alpha.$$

Proof. It follows from (4.9) that

$$\begin{aligned} \langle \rho_\alpha^*(\beta) - \beta, v_i \rangle &= \langle \beta, \rho_{\alpha^*}(v_i) \rangle - \langle \beta, v_i \rangle = \langle \beta, v_j \rangle - \langle \beta, v_i \rangle \\ \langle \rho_\alpha^*(\beta) - \beta, v_j \rangle &= \langle \beta, \rho_{\alpha^*}(v_j) \rangle - \langle \beta, v_j \rangle = \langle \beta, v_i \rangle - \langle \beta, v_j \rangle \\ \langle \rho_\alpha^*(\beta) - \beta, v_k \rangle &= \langle \beta, \rho_{\alpha^*}(v_k) \rangle - \langle \beta, v_k \rangle = 0 \quad \text{for } k \neq i, j. \end{aligned}$$

This together with (4.8) shows that the three terms in the lemma take a same value on each v_ℓ . Since v_ℓ 's span $H_2(BT)$, the desired identity in the lemma follows. \square

Let $a_{\beta, \alpha}$ be the constant defined by

$$(4.10) \quad \rho_\alpha^*(\beta) = \beta - a_{\beta, \alpha} \alpha.$$

By Lemma 4.4 we have

$$(4.11) \quad a_{\beta, \alpha} = \frac{\langle \beta, v_i \rangle - \langle \beta, v_j \rangle}{\langle \alpha, v_i \rangle} = \frac{\langle \beta, v_j \rangle - \langle \beta, v_i \rangle}{\langle \alpha, v_j \rangle}.$$

The following is well-known (see [9, 9.4] but $a_{\beta, \alpha}$ is denoted $\langle \beta, \alpha \rangle$ in the book):

- (1) $a_{\beta, \alpha}$ is an integer,
- (2) $a_{\beta, \alpha} \neq 0$ if and only if $a_{\alpha, \beta} \neq 0$,
- (3) $0 \leq a_{\beta, \alpha} a_{\alpha, \beta} \leq 3$ if $\beta \neq \pm \alpha$

We set

$$N_\alpha := |\langle \alpha, v_i \rangle| = |\langle \alpha, v_j \rangle|.$$

We say that α and β are *joined* if $\rho_\alpha^*(\beta) \neq \beta$ (i.e. $a_{\beta, \alpha} \neq 0$). By (2) above, $\rho_\alpha^*(\beta) \neq \beta$ if and only if $\rho_\beta^*(\alpha) \neq \alpha$.

Lemma 4.5. *If α and β are joined, then $N_\alpha = N_\beta$.*

Proof. Since $N_{-\alpha} = N_\alpha$, we may assume $\beta \neq \pm \alpha$. Then $a_{\beta, \alpha}$ is a non-zero integer, so it follows from (4.11) that $|a_{\beta, \alpha}| \geq 2$ if $N_\alpha \neq N_\beta$ (note that since $\beta \neq \pm \alpha$, either $\langle \beta, v_i \rangle$ or $\langle \beta, v_j \rangle$ is 0 by Lemma 4.3). Changing the role of α and β , we also have that $|a_{\alpha, \beta}| \geq 2$ if $N_\alpha \neq N_\beta$. But this contradicts the above fact (3) that $0 \leq a_{\beta, \alpha} a_{\alpha, \beta} \leq 3$ if $\beta \neq \pm \alpha$. \square

$\Delta(G)$ decomposes into a direct sum of irreducible root systems. Since we are concerned with the isomorphism type of $\Delta(G)$ as a root system, we may assume that $N_\alpha = 1$ for any α by Lemma 4.5.

5. THE ROOT SYSTEM OF A MOMENT POLYTOPE

Remember that the correspondence from symplectic toric manifolds to their moment polytopes (which are non-singular) is one-to-one. Motivated by the observation made in Section 4, we make the following definition.

Definition. For a non-singular polytope P described in (3.7), we define

$$R(P) := \{\alpha \in H^2(BT) \mid \langle \alpha, v_i \rangle = 1, \langle \alpha, v_j \rangle = -1 \text{ for some } i, j, \\ \text{and } \langle \alpha, v_k \rangle = 0 \text{ for } k \neq i, j\},$$

and call it the *root system* of P . (It will be proved below that $R(P)$ is actually a root system.)

Remark. The root system $R(P)$ depends only on the v_i 's and not on the constants a_i 's used to describe the moment polytope P in (3.7).

Example. We identify $H_2(BT)$ with \mathbb{Z}^n and denote by $\{e_i\}_{i=1}^n$ the standard basis of \mathbb{Z}^n and by $\{e_i^*\}_{i=1}^n$ the basis of $(\mathbb{Z}^n)^*$ dual to $\{e_i\}_{i=1}^n$. Remember that m is the number of facets of P .

(1) Let $m = n + 1$ and take $v_i = e_i$ for $1 \leq i \leq n$ and $v_{n+1} = -\sum_{i=1}^n e_i$. Then

$$R(P) = \{\pm e_i^* \ (1 \leq i \leq n), \ \pm(e_i^* - e_j^*) \ (1 \leq i < j \leq n)\}$$

and this is a root system of type A_n . Note that the manifold M is the complex projective space $\mathbb{C}P^n$ of complex dimension n in this case.

(2) Let $n = 2$, $m = 4$ and take $v_1 = e_1$, $v_2 = e_2$, $v_3 = -e_1 + ae_2$ and $v_4 = -e_2$ where a is an arbitrary integer. If $a = 0$, then P is a rectangle and

$$R(P) = \{\pm e_1^*, \pm e_2^*\}$$

which is of type $A_1 \times A_1$, and if $a \neq 0$, then P is a trapezoid with two right angle corners and

$$R(P) = \{\pm e_1^*\}$$

which is of type A_1 . Note that the manifold M is a Hirzebruch surface in this case.

(3) If $n = 2$ and $m \geq 5$, then one easily checks that $R(P)$ is empty.

For $\alpha \in R(P)$ with $\langle \alpha, v_i \rangle = 1$ and $\langle \alpha, v_j \rangle = -1$, we define a reflection r_α on $H_2(BT)$ by

$$r_\alpha(v) := v - \langle \alpha, v \rangle (v_i - v_j)$$

which interchanges v_i and v_j and fixes v_k 's for $k \neq i, j$, and define its dual reflection r_α^\vee on $H^2(BT)$ by

$$\langle r_\alpha^\vee(\beta), v \rangle := \langle \beta, r_\alpha(v) \rangle = \langle \beta, v - \langle \alpha, v \rangle (v_i - v_j) \rangle.$$

This shows that

$$(5.1) \quad r_\alpha^\vee(\beta) = \beta - (\langle \beta, v_i \rangle - \langle \beta, v_j \rangle) \alpha.$$

In particular, $r_\alpha^\vee(\pm\alpha) = \mp\alpha$. One can easily check that r_α^\vee preserves $R(P)$. Comparing Lemma 4.4 with (5.1) and noting that we may assume $N_\alpha = 1$ for any α , we see that ρ_α^* agrees with r_α^\vee and hence we obtain the following.

Proposition 5.1. *$R(P)$ is a root system and if P is the moment polytope associated with a symplectic toric manifold (M, ω) and G is a compact Lie subgroup of $\text{Symp}(M, \omega)$ containing the torus T , then the root system $\Delta(G)$ of G is a subsystem of $R(P)$.*

We define a symmetric scalar product $(\ , \)$ on $H^2(BT)$ by

$$(5.2) \quad (\beta, \gamma) := \sum_{\ell=1}^m \langle \beta, v_\ell \rangle \langle \gamma, v_\ell \rangle.$$

One easily sees from the definition of $R(P)$ that

$$(5.3) \quad \begin{aligned} (\alpha, \alpha) &= 2 && \text{for } \alpha \in R(P), \\ (\alpha, \beta) &= 0 \text{ or } \pm 1 && \text{for } \alpha, \beta \in R(P) \text{ with } \beta \neq \pm\alpha. \end{aligned}$$

The group generated by the reflections r_α^\vee ($\alpha \in R(P)$) is called the *Weyl group* of $R(P)$.

Lemma 5.2. *The scalar product $(\ , \)$ is invariant under the Weyl group of $R(P)$.*

Proof. It suffices to check that $(r_\alpha^\vee(\beta), r_\alpha^\vee(\gamma)) = (\beta, \gamma)$ for $\alpha \in R(P)$ and $\beta, \gamma \in H^2(BT)$. By definition there are $i, j \in [m]$ such that $\langle \alpha, v_i \rangle = 1$, $\langle \alpha, v_j \rangle = -1$ and $\langle \alpha, v_k \rangle = 0$ for $k \neq i, j$. We denote $\langle \eta, v_\ell \rangle$ by η_ℓ for $\eta \in H^2(BT)$. Then since $(\alpha, \eta) = \eta_i - \eta_j$, it follows from (5.1) that

$$\begin{aligned} (r_\alpha^\vee(\beta), r_\alpha^\vee(\gamma)) &= (\beta - (\beta_i - \beta_j)\alpha, \gamma - (\gamma_i - \gamma_j)\alpha) \\ &= (\beta, \gamma) - (\beta_i - \beta_j)(\alpha, \gamma) - (\gamma_i - \gamma_j)(\beta, \alpha) + (\beta_i - \beta_j)(\gamma_i - \gamma_j)(\alpha, \alpha) \\ &= (\beta, \gamma) - (\beta_i - \beta_j)(\gamma_i - \gamma_j) - (\gamma_i - \gamma_j)(\beta_i - \beta_j) + 2(\beta_i - \beta_j)(\gamma_i - \gamma_j) \\ &= (\beta, \gamma), \end{aligned}$$

proving the lemma. \square

For $\alpha, \beta \in R(P)$ we define an integer $a_{\beta, \alpha}$ by

$$(5.4) \quad r_\alpha^\vee(\beta) = \beta - a_{\beta, \alpha} \alpha.$$

similarly to (4.10). If $\langle \alpha, v_i \rangle = 1$ and $\langle \alpha, v_j \rangle = -1$, then

$$(5.5) \quad a_{\beta, \alpha} = \langle \beta, v_i \rangle - \langle \beta, v_j \rangle$$

by (5.1). Another description of $a_{\beta, \alpha}$ is the following.

Lemma 5.3. $a_{\beta,\alpha} = (\alpha, \beta)$ for $\alpha, \beta \in R(P)$. In particular, $a_{\beta,\alpha} = a_{\alpha,\beta}$.

Proof. Since $r_\alpha^\vee(\alpha) = -\alpha$ and r_α^\vee is of order 2, it follows from Lemma 5.2 that

$$(\alpha, r_\alpha^\vee(\beta) + \beta) = (r_\alpha^\vee(\alpha), r_\alpha^\vee(r_\alpha^\vee(\beta) + \beta)) = (-\alpha, \beta + r_\alpha^\vee(\beta))$$

and hence $(\alpha, r_\alpha^\vee(\beta) + \beta) = 0$. This together with (5.3) and (5.4) implies the lemma. \square

Theorem 5.4. Any irreducible subsystem of $R(P)$ is of type A.

Proof. Let Φ be an irreducible subsystem of $R(P)$. The Cartan matrix $C(\Phi)$ of Φ is $(a_{\beta,\alpha})$ where α and β run over elements in a basis of Φ . The diagonal entries of $C(\Phi)$ are all 2 by (5.3) and $C(\Phi)$ is symmetric by Lemma 5.3. Therefore, Φ must be either of type A, D or E (see [9, p.59]).

Suppose that Φ is of type D or E. Then there are elements $\alpha, \beta, \gamma, \delta$ in the basis of Φ such that

$$(5.6) \quad a_{\beta,\alpha} = a_{\gamma,\alpha} = a_{\delta,\alpha} = -1.$$

As before, let v_i, v_j be the elements such that $\langle \alpha, v_i \rangle = 1$ and $\langle \alpha, v_j \rangle = -1$. It follows from (5.5) and (5.6) that the values which β, γ, δ take on v_i and v_j must be either $(-1, 0)$ or $(0, 1)$. Therefore two of β, γ, δ , say β and γ , must take the same values on v_i and v_j , say $(0, 1)$. (The same argument below will work for $(-1, 0)$.) Let v_k be the other element on which β takes a non-zero value. Then $r_\beta(v_j) = v_k$ and since $\langle \beta, v_j \rangle = 1$, we have $\langle \beta, v_k \rangle = -1$. Moreover, since $\langle \gamma, v_j \rangle = 1$ and $\beta \neq \gamma$, we have $\langle \gamma, v_k \rangle = 0$. Therefore

$$\begin{aligned} \langle r_\beta^\vee(\gamma) - \gamma, v_j \rangle &= \langle \gamma, r_\beta(v_j) \rangle - \langle \gamma, v_j \rangle = \langle \gamma, v_k \rangle - \langle \gamma, v_j \rangle = -1 \\ \langle r_\beta^\vee(\gamma) - \gamma, v_k \rangle &= \langle \gamma, r_\beta(v_k) \rangle - \langle \gamma, v_k \rangle = \langle \gamma, v_j \rangle - \langle \gamma, v_k \rangle = 1. \end{aligned}$$

Since β takes 1 on v_j and -1 on v_k , the above shows that $r_\beta^\vee(\gamma) - \gamma = -\beta$ and hence $a_{\gamma,\beta} = 1$ by (5.4). However $a_{\gamma,\beta}$ must be non-positive because β and γ are in the basis of Φ and $\beta \neq \gamma$. This is a contradiction. Thus Φ is neither of type D nor E and hence of type A. \square

We conclude this section with the following corollary which follows from Proposition 5.1 and Theorem 5.4.

Corollary 5.5. If G is a compact Lie subgroup of $\text{Symp}(M, \omega)$ containing the torus T , then any irreducible factor of $\Delta(G)$ is of type A.

6. CONNECTED MAXIMAL COMPACT LIE SUBGROUP OF $\text{Symp}(M, \omega)$

In this section we shall observe that the equality $\Delta(G) = R(P)$ is attained for some compact *connected* Lie subgroup G of $\text{Symp}(M, \omega)$.

As discussed in Section 2, we may think of M as $\mathcal{Z}_P / \text{Ker } \mathcal{V}$ and ω as the form induced from the standard form ω_0 on \mathbb{C}^m , where

$$(6.1) \quad \mathcal{Z}_P = \{z \in \mathbb{C}^m \mid \sum_{i=1}^m \left(\frac{1}{2}|z_i|^2 + a_i\right)\mu_i = 0\}$$

from (3.10) and

$$(6.2) \quad \text{Ker } \mathcal{V} = \{(g_1, \dots, g_m) \in (S^1)^m \mid \prod_{i=1}^m g_i^{\langle u, v_i \rangle} = 1 \text{ for } \forall u \in H^2(BT)\}$$

from (2.13) through the identification $H^2(BT) = \mathbb{Z}^n$ discussed in Section 3.

Lemma 6.1. *Let α be an element of $R(P)$ such that $\langle \alpha, v_i \rangle = 1$, $\langle \alpha, v_j \rangle = -1$ and $\langle \alpha, v_k \rangle = 0$ for $k \neq i, j$. Then $\mu_i = \mu_j$ in (6.1) and $g_i = g_j$ in (6.2).*

Proof. It follows from (3.3) that $\pi^*(\alpha) = \tau_i - \tau_j$. Applying ι^* to the both sides of this identity, we get the former identity in the lemma because $\iota^* \circ \pi^* = 0$ by (3.9) and $\mu_\ell = \iota^*(\tau_\ell)$ by (3.5). If we take the α as u in (6.2), then the condition $\prod_{i=1}^m g_i^{\langle u, v_i \rangle} = 1$ reduces to $g_i g_j^{-1} = 1$ and this proves the latter statement in the lemma. \square

The purpose of this section is to prove the following.

Proposition 6.2. *There is a closed connected subgroup \tilde{G} of the unitary group $U(m)$ which leaves \mathcal{Z}_P invariant and contains $\text{Ker } \mathcal{V}$ in its center (so that the action of \tilde{G} induces an effective action of $\tilde{G} / \text{Ker } \mathcal{V}$ on $M = \mathcal{Z}_P / \text{Ker } \mathcal{V}$) and $\Delta(\tilde{G} / \text{Ker } \mathcal{V}) = R(P)$.*

Proof. Let Φ be an irreducible factor of $R(P)$. It is of type A by Theorem 5.4. Suppose that the rank of Φ is $r - 1$. Then it follows from Lemma 6.1 that there is a subset $I(\Phi) := \{i_1, \dots, i_r\}$ of $[m]$ such that $\mu_{i_1} = \dots = \mu_{i_r}$ and $g_{i_1} = \dots = g_{i_r}$ for $g = (g_1, \dots, g_m) \in \text{Ker } \mathcal{V}$. Therefore the action of $U(m)$ on \mathbb{C}^m restricted to the subgroup

$$U(\Phi) := \{(x_{ij}) \in U(m) \mid x_{ij} = \delta_{ij} \text{ unless both } i \text{ and } j \text{ are in } I(\Phi)\},$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise, leaves \mathcal{Z}_P invariant and $U(\Phi)$ commutes with $\text{Ker } \mathcal{V}$. We note that the root system of $U(\Phi)$ is (isomorphic to) Φ .

Now we decompose $R(P)$ into sum of irreducible factors Φ_1, \dots, Φ_s . Then the subsets $I(\Phi_1), \dots, I(\Phi_s)$ of $[m]$ are disjoint. We consider the subgroup \tilde{G} of $U(m)$ generated by $\prod_{i=1}^s U(\Phi_i)$ and (the diagonal subgroup) $(S^1)^m$. Since $(S^1)^m$ contains $\text{Ker } \mathcal{V}$, so does \tilde{G} . It follows from the observation above that \tilde{G} commutes with $\text{Ker } \mathcal{V}$ and the action of $U(m)$ on \mathbb{C}^m restricted to \tilde{G} leaves \mathcal{Z}_P invariant so that the action descends to an effective action of $\tilde{G}/\text{Ker } \mathcal{V}$ on M . Since the action of \tilde{G} on \mathcal{Z}_P preserves the standard form $\omega_0|_{\mathcal{Z}_P}$, the induced action of $\tilde{G}/\text{Ker } \mathcal{V}$ on M preserves the form ω on M , see (2.14). Finally, $\Delta(\tilde{G}/\text{Ker } \mathcal{V}) = R(P)$ by construction. \square

7. AUTOMORPHISMS OF A MOMENT POLYTOPE

Since the dual of the Lie algebra of T can be naturally identified with $H^2(BT; \mathbb{R})$, we think that the moment map μ associated with (M, ω) takes values in $H^2(BT; \mathbb{R})$ and $P = \mu(M)$. When $g \in \text{Symp}(M, \omega)$ normalizes the torus T , we associated the ρ_g of $\text{Aut}(T)$ to g in Section 4, where

$$(7.1) \quad \rho_g(t) = gtg^{-1} \quad \text{for } t \in T$$

and $\text{Aut}(T)$ denotes the group of automorphisms of T . The moment map associated to (M, ω) with the T -action twisted by ρ_g is given by $\rho_g^* \circ \mu$, so the image of M by the map is $\rho_g^*(P)$. Since g preserves the form ω , the images of M by μ and $\rho_g^* \circ \mu$ are congruent modulo parallel translations in $H^2(BT; \mathbb{R})$. Motivated by this observation, we define

$$\text{Aut}(P) := \{\rho \in \text{Aut}(T) \mid \rho^*(P) \equiv P\}$$

where \equiv denotes congruence modulo parallel translations in $H^2(BT; \mathbb{R})$.

Remark. As remarked before, the root system $R(P)$ depends only on the v_i 's and not on the constants a_i 's used to define the moment polytope P in (2.1) or (3.7). However $\text{Aut}(P)$ actually depends on the a_i 's. For instance, $\text{Aut}(P)$ for a square P is (a dihedral group) of order 8 while $\text{Aut}(P)$ for a (non-square) rectangle P is of order 4.

The correspondence $g \rightarrow \rho_g$ defines a homomorphism

$$(7.2) \quad \mathcal{D}: N_G(T) \rightarrow \text{Aut}(P)$$

where G is any subgroup of $\text{Symp}(M, \omega)$ containing T (e.g. G may be the entire group $\text{Symp}(M, \omega)$) and $N_G(T)$ denotes the normalizer of T in G . If $g \in T$, then ρ_g is the identity; so T is in the kernel of \mathcal{D} .

Lemma 7.1. *If G is a compact Lie subgroup of $\text{Symp}(M, \omega)$ containing the torus T , then the kernel of \mathcal{D} is exactly T .*

Proof. We note that $g \in N_G(T)$ permutes the characteristic submanifolds M_i 's of M . Suppose that $g \in N_G(T)$ is in the kernel of \mathcal{D} . Then g maps M_i to itself for each i . Let x be a T -fixed point in M . Then $x = \bigcap_{i \in I} M_i$ for some $I \in [m]$ with cardinality n , so that x is fixed by g . We decompose the tangent space $\tau_x M$ of M at x into

$$\tau_x M = \bigoplus_{i \in I} \tau_x M / \tau_x M_i.$$

The differential $dg: \tau_x M \rightarrow \tau_x M$ preserves each real 2-dimensional eigenspace $\tau_x M / \tau_x M_i$ since g fixes x and maps M_i to itself for each i . The symplectic form ω determines an orientation on $\tau_x M / \tau_x M_i$ for each i and dg preserves the orientation on $\tau_x M / \tau_x M_i$ since g preserves the form ω .

Since G is compact, there exists a G -invariant Riemannian metric on M so that we may assume that dg is an orthogonal transformation on $\tau_x M / \tau_x M_i$ but since dg preserves the orientation on it, dg on $\tau_x M / \tau_x M_i$ is a rotation and hence there exists $t \in T$ such that $dg = dt$, i.e. $d(gt^{-1})$ is the identity on $\tau_x M$. On the other hand, since gt^{-1} is contained in G and G is compact, the fixed point set of gt^{-1} is a closed submanifold of M . The connected component of this submanifold containing x is of codimension 0 because $d(gt^{-1})$ is the identity on $\tau_x M$. Since M is connected, the connected component must agree with M and this means $g = t$, proving the lemma. \square

Corollary 7.2. *Let G be a compact Lie subgroup of $\text{Symp}(M, \omega)$ containing the torus T and let G^0 be the identity component of G . Then*

$$G/G^0 \cong \mathcal{D}(N_G(T))/\mathcal{D}(N_{G^0}(T)) \subset \text{Aut}(P)/\mathcal{D}(N_{G^0}(T))$$

where \mathcal{D} is the map in (7.2) and $N_{G^0}(T)$ is the normalizer of T in G^0 .

Proof. Since T is a maximal torus of G and maximal tori in G are conjugate to each other because G is compact, the inclusion $N_G(T) \rightarrow G$ induces an isomorphism

$$(7.3) \quad N_G(T)/N_{G^0}(T) \cong G/G^0.$$

This fact together with Lemm 7.1 implies the corollary. \square

We shall construct a cross section of the homomorphism \mathcal{D} in (7.2) when $G = \text{Symp}(M, \omega)$. We recall the description (3.7) of P :

$$(7.4) \quad P = \{u \in H^2(BT; \mathbb{R}) \mid \langle u, v_i \rangle \geq a_i \quad (i = 1, \dots, m)\}.$$

Let $\rho \in \text{Aut}(P)$. Since $\rho^*(P) \equiv P$, we have

$$\rho^*(P) = P + u_0 \quad \text{for some } u_0 \in H^2(BT; \mathbb{R}).$$

On the other hand, we have

$$\rho^*(P) = \{\rho^*(u) \in H^2(BT; \mathbb{R}) \mid \langle u, v_i \rangle \geq a_i \quad (i = 1, \dots, m)\}$$

by definition and this can be rewritten as

$$(7.5) \quad \rho^*(P) = \{u \in H^2(BT; \mathbb{R}) \mid \langle u, \rho_*^{-1}(v_i) \rangle \geq a_i \quad (i = 1, \dots, m)\}.$$

Since $\rho^*(P) \equiv P$, it follows from (7.4) and (7.5) that there exists a permutation σ on $[m]$ such that

$$(7.6) \quad \rho_*^{-1}(v_i) = v_{\sigma(i)} \text{ for any } i \in [m]$$

so that

$$\rho^*(P) = \{u \in H^2(BT; \mathbb{R}) \mid \langle u, v_{\sigma(i)} \rangle \geq a_i \quad (i = 1, \dots, m)\}.$$

Therefore

$$\begin{aligned} \langle u, v_{\sigma(i)} \rangle \geq a_{\sigma(i)} &\iff u \in P \\ &\iff u + u_0 \in \rho^*(P) \\ &\iff \langle u + u_0, v_{\sigma(i)} \rangle \geq a_i \\ &\iff \langle u, v_{\sigma(i)} \rangle \geq a_i - \langle u_0, v_{\sigma(i)} \rangle \end{aligned}$$

and this shows that

$$(7.7) \quad a_{\sigma(i)} = a_i - \langle u_0, v_{\sigma(i)} \rangle \quad \text{for any } i.$$

We make one more observation on the permutation σ .

Lemma 7.3. *There is a ring automorphism of $H^*(M)$ sending μ_i to $\mu_{\sigma(i)}$ for each i .*

Proof. Since σ is induced from the automorphism ρ of P , we note that $\cap_{i \in I} P_i = \emptyset$ if and only if $\cap_{j \in \sigma(I)} P_j = \emptyset$ for $I \subset [m]$. Furthermore we note that $\cap_{i \in I} P_i = \emptyset$ if and only if $\cap_{i \in I} M_i = \emptyset$. Therefore, Lemma 3.1 ensures that sending τ_i to $\tau_{\sigma(i)}$ for each i induces a ring automorphism of $H_T^*(M)$, which we denote by f .

Applying f to the both sides of (3.3), we have

$$(7.8) \quad f(\pi^*(u)) = \sum_{i=1}^m \langle u, v_i \rangle f(\tau_i) = \sum_{i=1}^m \langle u, v_i \rangle \tau_{\sigma(i)},$$

while it follows from (3.3) applied to $\rho^*(u)$ instead of u that we have

$$\begin{aligned} (7.9) \quad \pi^*(\rho^*(u)) &= \sum_{i=1}^m \langle \rho^*(u), v_i \rangle \tau_i = \sum_{i=1}^m \langle \rho^*(u), v_{\sigma(i)} \rangle \tau_{\sigma(i)} \\ &= \sum_{i=1}^m \langle u, \rho_*(v_{\sigma(i)}) \rangle \tau_{\sigma(i)} = \sum_{i=1}^m \langle u, v_i \rangle \tau_{\sigma(i)} \end{aligned}$$

where we used (7.6) at the last identity. Comparing (7.8) with (7.9), we obtain the identity $f(\pi^*(u)) = \pi^*(\rho^*(u))$ for any $u \in H^2(BT)$ and this shows that the ring automorphism f of $H_T^*(M)$ preserves the subalgebra $\pi^*(H^*(BT))$. Therefore f induces a ring automorphism \bar{f} of $H^*(M)$ by Proposition 3.3. Since $f(\tau_i) = \tau_{\sigma(i)}$ and $\mu_i = \iota^*(\tau_i)$ by (3.5), we have $\bar{f}(\mu_i) = \mu_{\sigma(i)}$ which proves the lemma. \square

We now define the unitary transformation F_ρ of \mathbb{C}^m by

$$(7.10) \quad F_\rho(z_1, \dots, z_m) := (z_{\sigma(1)}, \dots, z_{\sigma(m)}).$$

It preserves \mathcal{Z}_P because

$$\begin{aligned} z \in \mathcal{Z}_P &\iff \sum \left(\frac{1}{2} |z_{\sigma(i)}|^2 + a_{\sigma(i)} \right) \mu_{\sigma(i)} = 0 \quad (\text{by (3.10)}) \\ &\iff \sum \left(\frac{1}{2} |z_{\sigma(i)}|^2 + a_i - \langle u_0, v_{\sigma(i)} \rangle \right) \mu_{\sigma(i)} = 0 \quad (\text{by (7.7)}) \\ &\iff \sum \left(\frac{1}{2} |z_{\sigma(i)}|^2 + a_i \right) \mu_{\sigma(i)} = 0 \quad (\text{by (2) in Proposition 3.3}) \\ &\iff \sum \left(\frac{1}{2} |z_{\sigma(i)}|^2 + a_i \right) \mu_i = 0 \quad (\text{by Lemma 7.3}) \\ &\iff F_\rho(z) \in \mathcal{Z}_P \quad (\text{by (3.10) and (7.10)}). \end{aligned}$$

Let ϕ be the automorphism of $(S^1)^m$ defined by

$$(7.11) \quad \phi(g_1, \dots, g_m) := (g_{\sigma(1)}, \dots, g_{\sigma(m)}).$$

Then the map F_ρ is ϕ -equivariant.

Lemma 7.4. *Let $\mathcal{V}: (S^1)^m \rightarrow T$ be the homomorphism in (2.11). Then $\mathcal{V} \circ \phi = \rho \circ \mathcal{V}$. In particular ϕ preserves $\text{Ker } \mathcal{V}$.*

Proof. Noting that $\rho(\lambda_v(g)) = \lambda_{\rho_*(v)}(g)$, we see from (7.11), (2.11) and (7.6) that

$$\begin{aligned} \mathcal{V}(\phi(g_1, \dots, g_m)) &= \mathcal{V}(g_{\sigma(1)}, \dots, g_{\sigma(m)}) = \prod_{i=1}^m \lambda_{v_i}(g_{\sigma(i)}) \\ &= \prod_{i=1}^m \lambda_{\rho_*(v_{\sigma(i)})}(g_{\sigma(i)}) = \rho\left(\prod_{i=1}^m \lambda_{v_{\sigma(i)}}(g_{\sigma(i)})\right) \\ &= \rho\left(\prod_{i=1}^m \lambda_{v_i}(g_i)\right) = \rho(\mathcal{V}(g_1, \dots, g_m)). \end{aligned}$$

This proves the lemma. \square

Since $M = \mathcal{Z}_P / \text{Ker } \mathcal{V}$ and F_ρ is ϕ -equivariant, F_ρ induces a diffeomorphism \bar{F}_ρ of M by Lemma 7.4. By definition F_ρ is a unitary transformation on \mathbb{C}^m , so \bar{F}_ρ preserves the symplectic form ω and hence $\bar{F}_\rho \in \text{Symp}(M, \omega)$.

Finally we need to prove the following.

Lemma 7.5. \bar{F}_ρ normalizes T and $\mathcal{D}(\bar{F}_\rho) = \rho$.

Proof. We view an element $g = (g_1, \dots, g_m)$ of $(S^1)^m$ as a diffeomorphism of $\mathcal{Z}_P \subset \mathbb{C}^m$. Then $F_\rho \circ g \circ F_\rho^{-1} = \phi(g)$. This identity descends to an identity

$$(7.12) \quad \bar{F}_\rho \circ \mathcal{V}(g) \circ \bar{F}_\rho^{-1} = \mathcal{V}(\phi(g)) \quad \text{in } \text{Symp}(M, \omega).$$

Since $T = \mathcal{V}((S^1)^m)$, the identity (7.12) shows that \bar{F}_ρ normalizes T .

Let $t \in T$. Then

$$(7.13) \quad t = \mathcal{V}(g) = \prod_{i=1}^m \lambda_{v_i}(g_i) \quad \text{for some } g \in (S^1)^m$$

where (2.11) is used for the latter identity. Using (7.13) together with the definition of \mathcal{D} (see also (7.1)), (7.12), (7.11) and (7.6), we have

$$\begin{aligned} (\mathcal{D}(\bar{F}_\rho))(t) &= \bar{F}_\rho \circ t \circ \bar{F}_\rho^{-1} = \bar{F}_\rho \circ \mathcal{V}(g) \circ \bar{F}_\rho^{-1} = \mathcal{V}(\phi(g)) \\ &= \mathcal{V}(g_{\sigma(1)}, \dots, g_{\sigma(m)}) = \prod_{i=1}^m \lambda_{v_i}(g_{\sigma(i)}) = \prod_{i=1}^m \lambda_{\rho_*(v_{\sigma(i)})}(g_{\sigma(i)}) \\ &= \rho\left(\prod_{i=1}^m \lambda_{v_{\sigma(i)}}(g_{\sigma(i)})\right) = \rho\left(\prod_{i=1}^m \lambda_{v_i}(g_i)\right) = \rho(\mathcal{V}(g)) = \rho(t) \end{aligned}$$

and this proves the latter statement in the lemma. \square

8. MAXIMAL COMPACT LIE SUBGROUP OF $\text{Symp}(M, \omega)$

The purpose of this section is to prove the following.

Theorem 8.1. *If a compact Lie subgroup G of $\text{Symp}(M, \omega)$ containing the torus T satisfies the following two conditions:*

- (1) $\Delta(G) = R(P)$, and
- (2) *the map \mathcal{D} in (7.2) is surjective,*

then G is maximal among compact Lie subgroups of $\text{Symp}(M, \omega)$ containing the torus T . Moreover, there is a compact Lie subgroup G_{\max} of $\text{Symp}(M, \omega)$ which satisfies the conditions (1) and (2) above.

Proof. What we prove for the former part of the theorem is that if a compact Lie subgroup H of $\text{Symp}(M, \omega)$ contains the G in the theorem, then $H = G$.

Since G is a subgroup of H , the root system $\Delta(G)$ of G is a subsystem of the root system $\Delta(H)$ of H . On the other hand, since H is a compact Lie subgroup of $\text{Symp}(M, \omega)$ containing T , $\Delta(H)$ is a subsystem of $R(P)$. Therefore, it follows from the condition (1) in the theorem that $\Delta(G) = \Delta(H)$ and this shows that $G^0 = H^0$ where the superscript 0 denotes the identity components as before.

Since G is a subgroup of H , $N_G(T)$ is a subgroup of $N_H(T)$. It follows from Lemma 7.1 and the condition (2) in the theorem that $N_G(T) = N_H(T)$ and this together with the isomorphism (7.3) for G and H implies $G = H$ because $G^0 = H^0$.

The proof of the latter part of the theorem is as follows. Let \tilde{G}_{\max} be the subgroup of $U(m)$ generated by \tilde{G} in Proposition 6.2 and F_ρ 's in (7.10) (regarded as elements of $U(m)$) for $\rho \in \text{Aut}(P)$. The identity component of \tilde{G}_{\max} is \tilde{G} . The action of \tilde{G}_{\max} on \mathbb{C}^m leaves \mathcal{Z}_P invariant and induces an effective action of $G_{\max} := \tilde{G}_{\max} / \text{Ker } \mathcal{V}$ on M preserving ω . The group G_{\max} contains the torus T and satisfies the two conditions in the theorem by Proposition 6.2 and Lemma 7.5. \square

Acknowledgment. I would like to thank Nigel Ray for his invitation to University of Manchester in summer of 2008 and stimulating discussions. This work is an outcome of a project with him. I also would like to thank Michael Wiemeler for explaining his work [12] which motivated the introduction of the root system $R(P)$. Finally I would like to thank Megumi Harada and Shintaro Kuroki for their helpful comments on an earlier version of the paper.

REFERENCES

- [1] F. Bosio and L. Meersseman, *Real quadrics in \mathbb{C}^n , complex manifolds and convex polytopes*, Acta Math. 197 (2006), 53–127.
- [2] V. M. Buchstaber and T. E. Panov, *Torus Actions and Their Applications in Topology and Combinatorics*, University Lecture, vol. 24, Amer. Math. Soc., Providence, R.I., 2002.
- [3] D. Cox, *The homogeneous coordinate ring of a toric variety*, J. Algebraic Geom. 4 (1995), 17–50.
- [4] T. Delzant, *Hamiltoniens périodiques et image convexe de l'application moment*, Bull. Soc. Math. France 116 (1988), 315–339.
- [5] M. Demazure, *Sous-groupes algébriques de rang maximum du groupe de Cremona*, Ann. Sci. École Norm. Sup. (4) 3 (1970), 507–588.
- [6] V. Guillemin, *Moment Maps and Combinatorial Invariants of Hamiltonian T^n -spaces*, Prog. in Math. 122, Birkhäuser, 1994.
- [7] A. Hattori and M. Masuda, *Theory of multi-fans*, Osaka J. Math. 40 (2003), 1–68.

- [8] Y. Karshon, L. Kessler and M. Pinsonnault, *A compact symplectic four-manifold admits only finitely many inequivalent toric actions*, J. of Symp. Geom. 5 (2007), 139-166.
- [9] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Math. 9, Springer-Verlag, 1970.
- [10] M. Masuda. *Unitary toric manifolds, multi-fans and equivariant index*, Tohoku Math. J. 51 (1999), 237–265.
- [11] T. Oda, *Convex Bodies and Algebraic Geometry. An Introduction to the Theory of Toric Varieties*, Ergeb. Math. Grenzgeb. (3), 15, Springer-Verlag, Berlin, 1988.
- [12] M. Wiemeler, *Torus manifolds with non-abelian symmetries*, preprint 2008.

DEPARTMENT OF MATHEMATICS, OSAKA CITY UNIVERSITY, SUMIYOSHI-KU,
OSAKA 558-8585, JAPAN.

E-mail address: masuda@sci.osaka-cu.ac.jp